Journal of Approximation Theory **102**, 243–262 (2000)
Article ID jath.1999.3412, available online at http://www.idealibrary.com on IDEAL®

Interpolation by Weak Chebyshev Spaces

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Communicated by Borislav Bojanov

Received October 20, 1998; accepted in revised form June 17, 1999

We present two characterizations of Lagrange interpolation sets for weak Chebyshev spaces. The first of them is valid for an arbitrary weak Chebyshev space U and is based on an analysis of the structure of zero sets of functions in U extending Stockenberg's theorem. The second one holds for all weak Chebyshev spaces that possess a locally linearly independent basis. © 2000 Academic Press

1. INTRODUCTION

Let U denote a finite-dimensional subspace of real valued functions defined on a totally ordered set K, for example, an arbitrary subset of $\mathbb{R}$.

A finite subset $T = \{t_1, ..., t_n\}$ of K, where $n = \dim U$, is called an *interpolation set* (*I-set*) w.r.t. U if for any given data $\{y_1, ..., y_n\}$ there exists a unique function $u \in U$ such that

$$u(t_i) = y_i, \quad i = 1, ..., n.$$

It is easy to see that T is an I-set w.r.t. U if and only if

$$\dim U_{|T} = n,$$

[1] The work of this author was supported by the Alexander von Humbold Foundation, under Research Fellowship.

0021-9045/00 $35.00

where $U_{|T} := \{u_{|T} : u \in U\}$. For a set of s points, $T = \{t_1, ..., t_s\} \subset K$, with $s < n$, we say that T is an *I*-set if $\dim U_{|T} = s$.

We are interested in describing *I*-sets w.r.t. U in the case when U is a *weak Chebyshev space* (*WT-space*), i.e., every $u \in U$ has at most $n-1$ sign changes. The primary example of a *WT*-space is the space of univariate polynomial splines, in which case all interpolation sets can be characterized by well-known Schoenberg–Whitney condition (see e.g. [12]). Extensions of Schoenberg–Whitney theorem to some classes of generalized spline spaces were proposed in [11, 13, 14].

Recently, some characterizations of *I*-sets w.r.t. weak Chebyshev spaces without any *a priori* assumption about "piecewise Haar" spline-like structure have been found. In [4] it was proved that Schoenberg–Whitney characterization in its "dimension form" holds true for a *WT*-space U if and only if $U_{|K'}$ is also a *WT*-space for all $K' \subset K$. This last property is satisfied, for example, if U is a weak Descartes space. In [2] the "support form" of Schoenberg–Whitney theorem has been shown to hold true for every *WT*-space that possesses a locally linearly independent weak Descartes basis. (See [7] for a review of various forms of Schoenberg–Whitney condition, especially in regard to their extendibility to multivariate splines.)

The purpose of this paper is twofold. In Section 2 we present a characterization of *I*-sets w.r.t. arbitrary weak Chebyshev spaces (Theorem 2.1), which does not involve any structural properties of U. Instead, the unions of the intervals $[t_i, t_{i+1}]$ between interpolation points are considered. This result relies on an extension of Stockenberg's theorem about zeros of functions in a *WT*-space (Theorem 2.4), which seems to be of independent interest. A (rather lengthy) proof of Theorem 2.4 is given in Section 4.

In Section 3 we generalize the above mentioned theorem of [2] and show that Schoenberg–Whitney characterization holds true for all weak Chebyshev spaces with a locally linearly independent basis. The proof involves an analysis of the relationship between *I*-sets and so-called *strong almost interpolation sets* as well as our previous results on almost interpolation [5, 9] and Theorem 2.1.

2. INTERPOLATION BY ARBITRARY WEAK CHEBYSHEV SPACES

We denote by $F(K)$ the linear space of all real valued functions defined on K and by $C(K)$ its subspace consisting of all continuous functions. For any $f \in F(K)$ and any subspace U of $F(K)$, let

$$Z(f) := \{t \in K : f(t) = 0\}, \qquad Z(U) := \bigcap_{f \in U} Z(f).$$

We need the following (somewhat unusual in the case $\beta < \alpha$) definition of the *closed interval* with endpoints $\alpha, \beta \in K$,

$$[\alpha, \beta] := \begin{cases} \{t \in K : \alpha \leqslant t \leqslant \beta\} & \text{if} \quad \alpha \leqslant \beta \\ \{t \in K : t \geqslant \alpha \text{ or } t \leqslant \beta\} & \text{if} \quad \beta < \alpha. \end{cases}$$

In the same way we define *open* and *halfopen intervals.*

The main result of this section reads as follows.

THEOREM 2.1. *Let U be an n-dimensional weak Chebyshev subspace of $F(K)$, and let $T = \{t_1, ..., t_n\} \subset K \setminus Z(U)$ such that $t_1 < \cdots < t_n$, and $t_{n+1} := t_1$. The following conditions are equivalent*:

(1) *T is an I-set w.r.t. U.*

(2) *For all $P \subset \{1, ..., n\}$,*

$$\text{card}\left(T \cap \bigcup_{i \in P} [t_i, t_{i+1}]\right) \leqslant \dim U_{|\bigcup_{i \in P} [t_i, t_{i+1}]}. \tag{2.1}$$

A simple example shows that this characterization is no longer valid if one omits the assumption that U is a weak Chebyshev space. Moreover, we conjecture that only for WT-spaces every set T satisfying condition (2) is an I-set.

EXAMPLE 2.2. Let $K = [0, 3] \subset \mathbb{R}$ and assume that $U = \text{span}\{u_1, u_2\}$ where $u_1 = 1$ on K and

$$u_2(t) = \begin{cases} 1 - t & \text{if} \quad 0 \leqslant t \leqslant 1 \\ 0 & \text{if} \quad 1 < t < 2 \\ t - 2 & \text{if} \quad 2 \leqslant t \leqslant 3. \end{cases}$$

Set $\tilde{u} = 1/2 u_1 - u_2$. Then it is obvious that $\tilde{u}$ has two sign changes at $t_1 = 1/2$ and $t_2 = 5/2$, respectively. This implies that U fails to be a weak Chebyshev space and, in particular, that $T = \{t_1, t_2\}$ fails to be an I-set w.r.t. U. On the other hand, T satisfies condition (2) of Theorem 2.1.

We will prove Theorem 2.1 at the end of this section as a consequence of a result about location of zeros of functions in WT-spaces. The following generalized notion of separation of zeros will be particular important for our analysis.

Suppose that $u \in U$ and $\tilde{Z} \subset Z(u)$ are given. We say that zeros $x_1, ..., x_m \in Z(u) \setminus \tilde{Z}$, where $x_1 < \cdots < x_m$, are *cyclically separated with respect to* $\tilde{Z}$ if for every $i \in \{1, ..., m\}$ there exists a subinterval

$$[y_{2i-1}, y_{2i}] \subset (x_i, x_{i+1}), \quad y_{2i-1}, y_{2i} \in K, \tag{2.2}$$

(we set $x_{m+1} := x_1$ if $i = m$) such that

$$u(y_j) \neq 0, \quad j = 1, ..., 2m, \tag{2.3}$$

and

$$(x_i, x_{i+1}) \cap \tilde{Z} \subset (y_{2i-1}, y_{2i}). \tag{2.4}$$

Note that $y_{2i-1} \leqslant y_{2i}$, $i = 1, ..., m-1$. For $i = m$ the following three cases can occur: (1) $x_m < y_{2m-1} \leqslant y_{2m}$, (2) $y_{2m-1} \leqslant y_{2m} < x_1$, and (3) $y_{2m} < x_1 < x_m < y_{2m-1}$.

If $\tilde{Z} \cap (x_i, x_{i+1}) = \emptyset$, then it is sufficient to have one point $y_{2i-1} = y_{2i}$, with $u(y_{2i-1}) \neq 0$. Otherwise, we need two different points y_{2i-1} and y_{2i} satisfying (2.4). If $\tilde{Z} = \emptyset$, then we simply say that $x_1, ..., x_m$ are *cyclically separated zeros* of u. Note that this definition requires that x_m and x_1 are also separated from each other by some points y_{2m-1} and y_{2m} (possibly equal) that lie outside $[x_1, x_m]$, which is the reason for the word "cyclically". If $\tilde{Z} = \emptyset$ and (2.2) and (2.3) are only satisfied for $i = 1, ..., m-1$ and $j = 1, ..., 2m-2$, respectively, then the zeros $x_1, ..., x_m$ are *separated* in usual sense.

We say that $x \in Z(u)$ is an *essential zero* of $u \in U$ if $x \notin Z(U)$.

A relationship between the number of separated essential zeros of functions $u \in U$ and the dimension n of U was found by Stockenberg [15]. We recall his theorem which has played an important role in characterizing continuous selections for metric projections (see e.g. [10]).

THEOREM 2.3. [15] *Suppose that U is an n-dimensional weak Chebyshev space.*

(1) *If there exists $u \in U$ with n separated essential zeros $x_1, ..., x_n$ such that $x_1 < \cdots < x_n$, then $u(t) = 0$ for all $t \in [x_n, x_1]$.*

(2) *No $u \in U$ has more than n separated essential zeros.*

Note that both statements of Theorem 2.3 are obviously contained in the following formulation: *no $u \in U$ has more than $n-1$ cyclically separated essential zeros.*

We generalize Theorem 2.3 as follows.

THEOREM 2.4. *Suppose that U is an n-dimensional weak Chebyshev subspace of $F(K)$, and let $\tilde{Z} \subset K$. If there exists $u \in U$ such that $\tilde{Z} \subset Z(u)$, and u has m essential zeros $x_1 < \cdots < x_m$ that are cyclically separated w.r.t. $\tilde{Z}$, then*

$$\dim U_{|\tilde{Z}} \leqslant n - m - 1. \tag{2.5}$$

The proof of Theorem 2.4 will be given in Section 4.

Let us see that Theorem 2.4 contains Theorem 2.3 as a special case. Indeed, if we take $\tilde{Z} = \emptyset$, then (2.5) gives the bound $m \leqslant n-1$ on the number m of cyclically separated essential zeros of u, which yields Theorem 2.3.

Moreover, in the situation of Theorem 2.3, (1), i.e., when $u \in U$ has n separated essential zeros $x_1 < \cdots < x_n$, we can deduce from (2.5) a slightly stronger statement by setting $\tilde{Z} := [x_n, x_1]$. Since $x_1, ..., x_n$ are separated in $[x_1, x_n]$, it clearly follows that $x_2, ..., x_{n-1}$ are cyclically separated w.r.t. $\tilde{Z}$. Thus, by (2.5), and in view of the assumption that $x_1 \notin Z(U)$, we have $\dim U_{|[x_n, x_1]} = 1$. Therefore, *under the hypotheses of Theorem* 2.3, (1), *not only u itself, but also every function $v \in U$ such that $v(x_1) = 0$ or $v(x_n) = 0$ necessarily satisfies $v(t) = 0$ for all $t \in [x_n, x_1]$*. Particularly, in the important special case when K contains its minimal and maximal elements, $a = \min K$ and $b = \max K$, we have the following corollary: *if* $\dim U_{|\{a, b\}} = 2$, *then no $u \in U$ has more than $n-1$ separated essential zeros*. This applies specifically to spline spaces and recovers their well-known property (see [12]).

We will see now that Theorem 2.1 immediately follows from Theorem 2.4.

Proof of Theorem 2.1. If T is an I-set, then $\operatorname{card}(T \cap K') \leqslant \dim U_{|K'}$ for *every* subset $K' \subset K$. Therefore, we only have to show that (2) implies (1). On the contrary, suppose that $T = \{t_1, ..., t_n\}$ satisfies (2), but fails to be an I-set. Hence, there exists a function $u \in U \setminus \{0\}$ such that

$$u(t_i) = 0, \qquad i = 1, ..., n.$$

We set

$$\tilde{P} = \{i : u_{|[t_i, t_{i+1}]} = 0\}, \qquad \tilde{Z} = \bigcup_{i \in \tilde{P}} [t_i, t_{i+1}].$$

Let

$$T \setminus \tilde{Z} = \{x_1, ..., x_m\}.$$

Then it is easy to see that $x_1, ..., x_m$ are essential zeros of u that are cyclically separated w.r.t. $\tilde{Z}$. Therefore, by (2.5), dim $U_{|\tilde{Z}} \leqslant n-m-1$. On the other hand, (2.1), with $P=\tilde{P}$, implies dim $U_{|\tilde{Z}} \geqslant n-m$, a contradiction. ∎

3. *WT*-SPACES WITH A LOCALLY LINEARLY INDEPENDENT BASIS

Throughout this section we assume that K is endowed with a topology consistent with the ordering. We denote by $\bar{M}$ and int M the closure and the interior, respectively, of any subset $M \subset K$. For every function $f \in F(K)$, we set

$$\operatorname{supp} f := \overline{\{x \in K : f(x) \neq 0\}}.$$

Let $\{u_1, ..., u_n\}$ be a system of functions in $F(K)$. The following notion of a locally linearly independent system which generalizes a well-known property of univariate B-splines has proven to be important in the problems of interpolation.

DEFINITION 3.1. We say that $\{u_1, ..., u_n\}$ is a *locally linearly independent system* (*LI-system*) if for any $t \in K$ and any neighborhood $B(t)$ of t there exists an open set B', with $t \in B' \subset B(t)$, such that the subsystem

$$\{u_i : B' \cap \operatorname{supp} u_i \neq \emptyset\}$$

is linearly independent on B'.

It has been shown in [9] that the above definition is equivalent to the standard definition of local linear independence by de Boor and Höllig [1], so that $\{u_1, ..., u_n\}$ is an *LI*-system if and only if for every open $B \subset K$, the condition

$$\sum_{i=1}^{n} a_i u_i(x) = 0, \qquad x \in B,$$

implies $a_i = 0$ for all i such that $B \cap \operatorname{supp} u_i \neq \emptyset$.

An important feature of an *LI*-system $\{u_1, ..., u_n\}$ is that it forms a least supported basis for its span (see Carnicer and Peña [3]).

Carnicer and Peña [2] have also shown that for a space of continuous functions on the real interval spanned by an *LI*-system satisfying weak Descartes property, interpolation sets can be characterized by Schoenberg–Whitney condition in support form. In order to formulate this result, we need the definition of a weak Descartes system. Recall that a matrix is said to be *totally positive* if all its minors are nonnegative.

DEFINITION 3.2. We say that a system of functions $\{u_1, ..., u_n\}$ in $F(K)$ is a *weak Descartes system* (*WD-system*) if the matrix $(u_i(t_j))_{i,j=1}^n$ is totally positive for all choices of $t_1, ..., t_n \in K$ such that $t_1 < \cdots < t_n$.

THEOREM 3.3. [2] *Suppose that $K = [a, b]$ is an interval of the real line $\mathbb{R}$, and $\{u_1, ..., u_n\} \subset C(K)$ is simultaneously an LI-system and WD-system. Let $T = \{t_1, ..., t_n\} \subset K$ such that $t_1 < \cdots < t_n$. The following conditions are equivalent*:

(1) *T is an I-set w.r.t. $U = \text{span}\{u_1, ..., u_n\}$.*

(2) $t_i \in \{x \in K : u_i(x) \neq 0\}$, $i = 1, ..., n$.

The main objective of this section is to provide a generalization of Theorem 3.3 in two directions. First, we relax the condition that $\{u_1, ..., u_n\}$ is a *WD*-system and show that the theorem essentially holds for every weak Chebyshev space possessing an *LI*-basis. Second, we allow K to be a general totally ordered set. Our main tools are Theorem 2.1 and some results of our previous research [8, 9] on almost interpolation by spaces with locally linearly independent bases.

DEFINITION 3.4. Let U be a finite-dimensional subspace of $F(K)$, $\dim U = n$. A set $T = \{t_1, ..., t_s\} \subset K$, $s \leqslant n$ is called an *almost interpolation set* (*AI-set*) w.r.t. U if for any system of neighborhoods B_i of t_i, $i = 1, ..., s$, there exist points $t_i' \in B_i$ such that $T' = \{t_1', ..., t_s'\}$ is an I-set w.r.t. U.

Next two theorems are valid for any topological space K.

THEOREM 3.5. [5] *Let U be a finite-dimensional subspace of $F(K)$, $\dim U = n$, and let $T = \{t_1, ..., t_s\} \subset K$, $s \leqslant n$. Then T is an AI-set w.r.t. U if and only if*

$$\dim U_{|B(T')} \geqslant \text{card } T', \qquad \textit{all open} \quad B(T') \supset T', \tag{3.1}$$

for every choice of a nonempty subset $T' \subset T$.

We note that condition (3.1) can be also written as

$$\text{l-dim}_{T'} U \geqslant \text{card } T',$$

where $\text{l-dim}_{T'} U$ denotes the *local dimension of U on T'*, i.e.,

$$\text{l-dim}_{T'} U := \inf\{\dim U_{|B} : T' \subset B, \ B \text{ open}\}.$$

THEOREM 3.6. [9] *Let* $\{u_1, ..., u_n\} \subset F(K)$ *be a locally linearly independent system and* $U = \text{span}\{u_1, ..., u_n\}$. *A finite set* $T = \{t_1, ..., t_s\} \subset K$, $s \leqslant n$, *is an AI-set w.r.t. U if and only if there exists some permutation* σ *of* $\{1, ..., n\}$ *such that*

$$t_i \in \text{supp } u_{\sigma(i)}, \qquad i = 1, ..., s.$$

Generally, for example if K is a domain in $\mathbb{R}^d$, $d > 1$, many almost interpolation sets fail to be *I*-sets. However, in our case of a totally ordered K the situation is much better. In fact, it is often enough to strengthen the condition of almost interpolation in the following obvious way, in order to get a characterization of *I*-sets.

DEFINITION 3.7. A set $T = \{t_1, ..., t_s\} \subset K$, $s \leqslant n$ is called a *strong AI-set* w.r.t. U if there exist neighborhoods B_i of t_i, $i = 1, ..., s$, such that $T' = \{t'_1, ..., t'_s\}$ is an *AI*-set w.r.t. U as soon as $t'_i \in B_i$, $i = 1, ..., s$.

If U includes only continuous functions on K, i.e., $U \subset C(K)$, then every *I*-set w.r.t. U is easily seen to be a strong *AI*-set.

Let now K be again a totally ordered set. We say that a point $t \in K$ has *V-property* if either t is an isolated point of K, or

$$t = \sup\{x \in K : x < t\} = \inf\{x \in K : x > t\}.$$

(The latter means, in particular, that both sets in the last display are nonempty.)

LEMMA 3.8. *Let U be a finite-dimensional subspace of* $F(K)$, $\dim U = n$, *and let* $T = \{t_1, ..., t_s\} \subset K$, $s \leqslant n$, *such that* $t_1 < \cdots < t_s$, *and* $t_{s+1} := t_1$. *Suppose that every point* t_i, $i = 1, ..., s$, *satisfies V-property. If T is a strong AI-set w.r.t. U, then for every* $P \subset \{1, ..., s\}$,

$$\text{card}\left(T \cap \bigcup_{i \in P} [t_i, t_{i+1}]\right) \leqslant \dim U_{|\bigcup_{i \in P}[t_i, t_{i+1}]}.$$

Proof. On the contrary, suppose that

$$\dim U_{|R_{\tilde{P}}} < \text{card}\,(T \cap R_{\tilde{P}}) \tag{3.2}$$

for some $\tilde{P} \subset \{1, ..., s\}$, where $R_{\tilde{P}} = \bigcup_{j \in \tilde{P}} [t_j, t_{j+1}]$. Then we also have

$$\dim U_{|\text{int } R_{\tilde{P}}} < \text{card}\,(T \cap R_{\tilde{P}}). \tag{3.3}$$

However, since T is a strong AI-set and every point t_i, $i=0, ..., s$, satisfies V-property, for each $t_j \in R_{\tilde{P}}$ we can find a point $t'_j \in \operatorname{int} R_{\tilde{P}}$ such that

$$T' = \{t'_1, ..., t'_s\}$$

is an AI-set (we set $t'_j = t_j$ when $t_j \notin R_{\tilde{P}}$). Thus,

$$\operatorname{card}(T' \cap \operatorname{int} R_{\tilde{P}}) = \operatorname{card}(T \cap R_{\tilde{P}}), \tag{3.4}$$

and, because of (3.3),

$$\dim U_{|\operatorname{int} R_{\tilde{P}}} < \operatorname{card}(T' \cap \operatorname{int} R_{\tilde{P}}),$$

which is impossible in view of Theorem 3.5. ∎

From this lemma and Theorem 2.1 we immediately get the following result describing relationship between I-sets and strong AI-sets w.r.t. a weak Chebyshev space.

THEOREM 3.9. *Let U be an n-dimensional weak Chebyshev subspace of $F(K)$, and let $T = \{t_1, ..., t_n\} \subset K \backslash Z(U)$. Suppose that every point t_i, $i = 1, ..., n$, satisfies V-property.*

(1) *If T is a strong AI-set w.r.t. U, then T is an I-set.*

(2) *Moreover, if $U \subset C(K)$, then the following conditions are equivalent*:

- *T is an I-set w.r.t. U.*
- *T is a strong AI-set w.r.t. U.*

The following example shows that Theorem 3.9 is not true in general if the points of T do not satisfy V-property.

EXAMPLE 3.10. Let $K = [-1, 1] \cup \{-2, 2\} \subset \mathbb{R}$ and assume that $U = \operatorname{span}\{u_1, u_2\}$ where $u_1(t) = t$, $t \in K$, and

$$u_2(t) = \begin{cases} 1 - t^2 & \text{if } \ t \in [-1, 1] \\ 0 & \text{if } \ t \in \{-2, 2\}. \end{cases}$$

It then follows that U is a weak Chebyshev space. Set $T = \{t_1, t_2\}$ where $t_1 = -1$ and $t_2 = 1$. Then in view of Theorem 3.5, T is an AI-set w.r.t. U. Moreover, T is a strong AI-set, since $T' = \{t'_1, t'_2\}$ is an AI-set for all $t_1 \leqslant t'_1 < t'_2 \leqslant t_2$. However, T fails to be an I-set w.r.t. U since $T \subset Z(u_2)$. It is also easy to see that both t_1 and t_2 fail to have V-property.

We now turn to the main subject of this section: characterization of I-sets for weak Chebyshev spaces with LI-basis.

THEOREM 3.11. *Let U be an n-dimensional weak Chebyshev subspace of $C(K)$, such that $U = \operatorname{span}\{u_1, ..., u_n\}$, where $\{u_1, ..., u_n\}$ is an LI-system, and let $T = \{t_1, ..., t_n\} \subset K \backslash Z(U)$. Suppose that every point t_i, $i = 1, ..., n$, satisfies V-property. Then T is an I-set w.r.t. U if and only if there exists some permutation σ of $\{1, ..., n\}$ such that*

$$t_i \in \operatorname{int} \operatorname{supp} u_{\sigma(i)}, \qquad i = 1, ..., n. \tag{3.5}$$

Proof. Let us first assume that (3.5) holds. We show that T is a strong AI-set w.r.t. U. It follows from Theorem 3.6 that T is an AI-set. Let $V_i := \operatorname{int} \operatorname{supp} u_{\sigma(i)}$, $i = 1, ..., n$. Then V_i is an open neighborhood of t_i, $i = 1, ..., n$, and again in view of Theorem 3.6, $T' = \{t'_1, ..., t'_n\}$ is an AI-set w.r.t. U for all $t'_i \in V_i$, $i = 1, ..., n$. This shows that T is a strong AI-set. It then follows from Theorem 3.9 that T is an I-set w.r.t. U. (This direction is even true without the assumption that $U \subset C(K)$.)

For the converse assume that T is an I-set w.r.t. U. We prove (3.5) by induction on n (where we do not use the hypothesis on U to be a weak Chebyshev space).

Let $n = 1$. Then $U = \operatorname{span}\{u_1\}$ and $T = \{t_1\} \subset K \backslash Z(u_1)$. Hence, $u_1(t_1) \neq 0$, and, since $u_1 \in C(K)$, $t_1 \in \operatorname{int} \operatorname{supp} u_1$.

Assume now that the statement is true up to $n-1$. Since $T = \{t_1, ..., t_n\}$ is an I-set w.r.t. U, it is also an AI-set, which, in view of Theorem 3.6, implies that there exists some permutation σ of $\{1, ..., n\}$ such that

$$t_i \in \operatorname{supp} u_{\sigma(i)}, \qquad i = 1, ..., n.$$

Without loss of generality assume that $\sigma(i) = i$, $i = 1, ..., n$. Suppose now that $t_1 \notin \operatorname{int} \operatorname{supp} u_1$. Then $u_1(t_1) = 0$ since $u_1 \in C(K)$.

Let $M := (u_i(t_j))_{i,j=1}^n$ and let M_{i1} denote the submatrix of M obtained by omitting the i-th row and the first column. Then

$$\det M = \sum_{i=1}^{n} (-1)^{i+1} u_i(t_1) \det M_{i1}.$$

Since T is an I-set, $\det M \neq 0$, which implies that $u_\ell(t_1) \neq 0$ and $\det M_{\ell 1} \neq 0$ for some $\ell \in \{2, ..., n\}$.

Hence, $\{t_2, ..., t_n\}$ is an I-set w.r.t. $\operatorname{span}\{u_1, ..., u_{\ell-1}, u_{\ell+1}, ..., u_n\}$. Applying the induction hypothesis to this situation we find a bijection $\tilde{\sigma}$ from $\{2, ..., n\}$ to $\{1, ..., \ell-1, \ell+1, ..., n\}$ such that

$$t_i \in \operatorname{int} \operatorname{supp} u_{\tilde{\sigma}(i)}, \quad i = 2, ..., n.$$

Moreover, $u_\ell(t_1) \neq 0$ implies that $t_1 \in \operatorname{int} \operatorname{supp} u_\ell$.

Therefore, setting

$$\sigma(i) := \begin{cases} \tilde{\sigma}(i) & \text{if} \quad i = 2, \dots, n, \\ \ell & \text{if} \quad i = 1, \end{cases}$$

we obtain the desired statement

$$t_i \in \operatorname{int} \operatorname{supp} u_{\sigma(i)}, \quad i = 1, \dots, n.$$

This completes the proof of Theorem 3.11. ∎

Example 3.10 also shows that V-property of points t_i is essential in the formulation of Theorem 3.11. Indeed, it is easy to see that the functions u_1, u_2 of Example 3.10 form an *LI*-system, and $T = \{-1, 1\}$ fails to be an *I*-set w.r.t. $U = \operatorname{span}\{u_1, u_2\}$ despite the fact that (3.5) holds.

If we now take $K = [a, b] \subset \mathbb{R}$, then V-property is satisfied for every $t \in K$ except $t = a$ or b. Therefore, the hypotheses of Theorem 3.11 do not allow T to include the endpoints of the interval $[a, b]$. In fact, some extra conditions at these points have to be imposed.

THEOREM 3.12. *Let U be an n-dimensional weak Chebyshev subspace of $F[a, b]$, such that $U = \operatorname{span}\{u_1, \dots, u_n\}$, where $\{u_1, \dots, u_n\}$ is an LI-system, and let $T = \{t_1, \dots, t_n\} \subset K \setminus Z(U)$. Suppose that there exists some permutation σ of $\{1, \dots, n\}$ such that*

(1) $t_i \in \operatorname{int} \operatorname{supp} u_{\sigma(i)}$, $i = 1, \dots, n$,

(2) $u_{\sigma(i)}(t_i) \neq 0$ *if* $t_i \in \{a, b\}$, *and*

(3) $u_{\sigma(i)}(t_i)\, u_{\sigma(j)}(t_j) - u_{\sigma(i)}(t_j)\, u_{\sigma(j)}(t_i) \neq 0$ *if* $t_i, t_j \in \{a, b\}$, $t_i \neq t_j$.

Then T is an I-set w.r.t. U.

Proof. Let us first consider the case $n = 1$. If $t_1 \in (a, b)$, then t_1 obviously satisfies V-property, and hence $T = \{t_1\}$ is an *I*-set w.r.t. U by Theorem 3.11. Otherwise, $u_1(t_1) \neq 0$ by (2), and T is an *I*-set again.

Suppose $n \geqslant 2$. If $T \subset (a, b)$, then every point t_i has V-property, and the statement follows from Theorem 3.11. However, in the case $T \cap \{a, b\} \neq \emptyset$ Theorem 3.11 is not applicable. Therefore, we argue as follows.

We first extend the interval $[a, b]$ to the open interval $\tilde{K} := (a - \varepsilon, b + \varepsilon)$ for some $\varepsilon > 0$. Then every $t_i \in T$, $i = 1, \dots, n$, obviously satisfies V-property (w.r.t. $\tilde{K}$). Moreover, extend each function u_i, $i = 1, \dots, n$, to a function $\tilde{u}_i \in F(\tilde{K})$ by

$$\tilde{u}_i(t) = \begin{cases} u_i(t) & \text{if} \quad t \in [a, b], \\ u_i(a) & \text{if} \quad t \in (a - \varepsilon, a), \\ u_i(b) & \text{if} \quad t \in (b, b + \varepsilon). \end{cases}$$

Then $\tilde{U} := \text{span}\{\tilde{u}_1, ..., \tilde{u}_n\}$ is again a weak Chebyshev space (while $\{\tilde{u}_1, ..., \tilde{u}_n\}$ is no longer an *LI*-system).

If T is a strong *AI*-set w.r.t. $\tilde{U}$, then, by Theorem 3.9, it is also an *I*-set w.r.t. $\tilde{U}$ and, in particular, w.r.t. U (since $U = \tilde{U}_{|[a,b]}$ and $T \subset [a, b]$). Thus, it suffices to show that T is a strong *AI*-set w.r.t. $\tilde{U}$.

To this end we consider sufficiently small open neighbourhoods V_i of t_i's, such that

$$V_i \subset \text{int supp } \tilde{u}_{\sigma(i)},$$

$$V_i \cap V_j = \emptyset \qquad \text{if} \quad i \neq j,$$

$$V_i \subset (a, b) \qquad \text{if} \quad t_i \in (a, b),$$

and take arbitrary points $\tilde{t}_i \in V_i$, $i = 1, ..., n$. We have to check that $\tilde{T} := \{\tilde{t}_1, ..., \tilde{t}_n\}$ is an *AI*-set w.r.t. $\tilde{U}$. In view of Theorem 3.5 this will follow if we prove that

$$\text{l-dim}_{T'} \tilde{U} \geqslant \text{card } T', \tag{3.6}$$

for every nonempty $T' \subset \tilde{T}$.

Suppose without loss of generality that $\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_n$. Let $T' = \{\tilde{t}_{i_1}, ..., \tilde{t}_{i_r}\} \subset \tilde{T}$. If $T' \subset (a, b)$, then Theorem 3.6 ensures that T' is an *AI*-set w.r.t. U since $\tilde{t}_{i_j} \in V_{i_j} \subset \text{supp } u_{\sigma(i_j)}$, $j = 1, ..., r$. Therefore,

$$\text{l-dim}_{T'} \tilde{U} = \text{l-dim}_{T'} U \geqslant r = \text{card } T'$$

by Theorem 3.5, and (3.6) holds.

Assume that $T' \backslash (a, b) \neq \emptyset$. Obviously, at most two points in T' may lie outside (a, b). We consider only the worst case $T' \backslash (a, b) = \{\tilde{t}_{i_1}, \tilde{t}_{i_r}\} = \{\tilde{t}_1, \tilde{t}_n\}$. (The other cases can be handled analogously.) Then necessarily $t_1 = a$, $t_n = b$. Set $\hat{T} := T' \backslash \{\tilde{t}_1, \tilde{t}_n\}$. Since $\hat{T} \subset (a, b)$, we have, as in the above,

$$\text{l-dim}_{\hat{T}} U \geqslant \text{card } \hat{T} = r - 2.$$

If $\text{l-dim}_{\hat{T}} U \geqslant r$, then

$$\text{l-dim}_{T'} \tilde{U} \geqslant \text{l-dim}_{\hat{T}} \tilde{U} = \text{l-dim}_{\hat{T}} U \geqslant r = \text{card } T',$$

and (3.6) follows. Otherwise, recall that by the definition of $\tilde{u}_i$ we have

$$\tilde{u}_{\sigma(1)}(\tilde{t}_1) = u_{\sigma(1)}(t_1), \qquad \tilde{u}_{\sigma(n)}(\tilde{t}_n) = u_{\sigma(n)}(t_n),$$

$$\tilde{u}_{\sigma(1)}(\tilde{t}_n) = u_{\sigma(1)}(t_n), \qquad \tilde{u}_{\sigma(n)}(\tilde{t}_1) = u_{\sigma(n)}(t_1),$$

and hence conditions (2) and (3) ensure that

$$\tilde{u}_{\sigma(1)}(\tilde{t}_1) \neq 0, \quad \tilde{u}_{\sigma(n)}(\tilde{t}_n) \neq 0, \quad \det \begin{pmatrix} \tilde{u}_{\sigma(1)}(\tilde{t}_1) & \tilde{u}_{\sigma(1)}(\tilde{t}_n) \\ \tilde{u}_{\sigma(n)}(\tilde{t}_1) & \tilde{u}_{\sigma(n)}(\tilde{t}_n) \end{pmatrix} \neq 0. \tag{3.7}$$

Moreover, by [9, Theorem 3.4], since $\{u_1, ..., u_n\}$ is an *LI*-system, we have

$$\text{l-dim}_{\hat{T}}\, U = \text{card}\, \{i = 1, ..., n : \hat{T} \cap \text{supp}\, u_i \neq \emptyset\}. \tag{3.8}$$

If $\text{l-dim}_{\hat{T}}\, U = r - 2$, then (3.8) implies that

$$\tilde{t} \notin \text{supp}\, u_{\sigma(1)} \cup \text{supp}\, u_{\sigma(n)}, \quad \text{all} \quad \tilde{t} \in \hat{T}.$$

Combining this with (3.7), we see that $\text{l-dim}_{T'} \tilde{U} \geqslant r$, and (3.6) holds. If $\text{l-dim}_{\hat{T}}\, U = r - 1$, then by (3.8),

$$\tilde{t} \notin \text{supp}\, u_{\sigma(i)}, \quad \text{all} \quad \tilde{t} \in \hat{T},$$

for at least one $i \in \{1, n\}$. Since $\tilde{u}_{\sigma(i)}(\tilde{t}_i) \neq 0$, $i \in \{1, n\}$, we again have $\text{l-dim}_{T'} \tilde{U} \geqslant r$, which completes the proof. ∎

It is easy to see that condition (3) of Theorem 3.12 is superfluous if $\{u_1, ..., u_n\} \subset C[a, b]$ is simultaneously an *LI*-system and *WD*-system, i.e., in the setting of Theorem 3.3. Indeed, in this case (3) is a consequence of (2) in view of the following lemma due to Carnicer and Peña.

LEMMA 3.13. [2] *Let $\{u_1, u_2\} \subset C[a, b]$ be simultaneously an LI-system and WD-system. If $u_1(a) \neq 0$ and $u_2(b) \neq 0$, then*

$$\det \begin{pmatrix} u_1(a) & u_1(b) \\ u_2(a) & u_2(b) \end{pmatrix} \neq 0.$$

Moreover, conditions (1) and (2) are now equivalent to

$$t_i \in \{x \in K : u_{\sigma(i)}(x) \neq 0\}, \qquad i = 1, ..., n,$$

which shows that Theorem 3.3 follows from Theorem 3.12.

4. PROOF OF THEOREM 2.4

On the contrary, suppose that

$$\dim U_{|\tilde{Z}} \geqslant n - m.$$

Then there exists $T=\{t_1, ..., t_{n-m}\} \subset \tilde{Z}$ such that

$$\dim U_{|T}=n-m,$$

and, since $x_1, ..., x_m$ are cyclically separated w.r.t. $\tilde{Z}$,

$$T \subset \bigcup_{i=1}^{m} (y_{2i-1}, y_{2i})$$

where $\{y_j\}_{j=1}^{2m}$ satisfy (2.2)–(2.4). We set

$$X:=\{x_1, ..., x_m\}, \qquad Y:=\{y_1, ..., y_{2m}\},$$

$$x_{m+1}:=x_1, \qquad y_{2m+1}:=y_1.$$

Let $j^* \in \{2m-2, 2m-1, 2m\}$ be a unique index such that

$$y_{j^*} > y_{j^*+1}.$$

We set

$$n_j:=\text{card }([y_j, y_{j+1}] \cap (X \cup T)), \qquad j=1, ..., 2m.$$

It is obvious that

$$\sum_{j=1}^{2m} n_j=n. \tag{4.1}$$

We now construct a function $v \in U$ such that $(X \cup T) \cap Z(v)=\varnothing$. Since $\dim U_{|T}=n-m=\text{card } T$, we interpolate on T as follows. Let $j \neq j^*$ and $T \cap [y_j, y_{j+1}] \neq \varnothing$. Then in view of (2.4), j is an odd number, which implies that $[y_j, y_{j+1}] \cap X=\varnothing$. Thus we have

$$T \cap [y_j, y_{j+1}]=\{t_{k_j} < \cdots < t_{k_j+n_j-1}\}, \qquad n_j \geqslant 1.$$

We require

$$\text{sign } v(t_{k_j+s})=(-1)^s \text{ sign } u(y_j), \qquad s=0, ..., n_j-1. \tag{4.2}$$

Consider now the index j^* and assume that $T \cap [y_{j^*}, y_{j^*+1}] \neq \varnothing$. In view of (2.4), it is quite clear that only the case $j^*=2m-1$; i.e., $y_{2m} < x_1$, $x_m < y_{2m-1}$ can occur. Moreover, it then follows that $n_{j^*}=\text{card}(T \cap [y_{j^*}, y_{j^*+1}]) \geqslant 1$ and there exists $p \in \{0, ..., n_{j^*}\}$ such that

$$T \cap [y_{j^*}, y_{j^*+1}]=\{t_{k_{j^*}}, ..., t_{k_{j^*}+n_{j^*}-1}\}$$

and

$$t_{k_{j^*}+p} < \cdots < t_{k_{j^*}+n_{j^*}-1} < y_{2m} < x_1$$
$$< \cdots < x_m < y_{2m-1} < t_{k_{j^*}} < \cdots < t_{k_{j^*}+p-1}.$$

This means that all points lie to the left of y_{2m} if $p=0$ and to the right of y_{2m-1} if $p=n_{j^*}$. If $p \neq 0$, we require

$$\operatorname{sign}\ v(t_{k_{j^*}+s}) = \begin{cases} (-1)^s \operatorname{sign} u(y_{2m-1}) & \text{if} \quad s=0, ..., p-1 \\ (-1)^{s+n-1} \operatorname{sign} u(y_{2m-1}) & \text{if} \quad s=p, ..., n_{j^*}-1. \end{cases} \tag{4.3}$$

If $p=0$, we require

$$\operatorname{sign}\ v(t_{k_{j^*}+s}) = (-1)^{s+n_{j^*}-1} \operatorname{sign} u(y_{2m}), \qquad s=0, ..., n_{j^*}-1. \tag{4.4}$$

Since $x_1, ..., x_m$ are essential zeros of u, we can apply [15, Lemma 2] and require

$$v(x_i) \neq 0, \qquad i=1, ..., m. \tag{4.5}$$

Thus, a function $v \in U$ with properties (4.2)–(4.5) must exist. In view of (2.3), we can find $\varepsilon > 0$ such that

$$|\varepsilon v(y_j)| < |u(y_j)|, \qquad j=1, ..., 2m. \tag{4.6}$$

We now show that at least one of the functions $u-\varepsilon v$, $u+\varepsilon v \in U$ has n sign changes on K contradicting the weak Chebyshev property of U.

To this end we determine a subset D of $\{1, ..., 2m\}$ as follows. We say that $j \in D$ if both $n_j \neq 0$ and

$$\operatorname{sign}\ u(y_j)\, u(y_{j+1}) = \begin{cases} (-1)^{n_j+1} & \text{if} \quad j \neq j^* \\ (-1)^{n_j+n} & \text{if} \quad j=j^*. \end{cases} \tag{4.7}$$

We now divide D into two subsets P and N as follows. Let $j \in D$. We say that $j \in P$ if either j is odd, or $j=2i$ and

$$\operatorname{sign}\ u(y_{2i})\, v(x_{i+1}) = \begin{cases} 1 & \text{if} \quad y_{2i} < x_{i+1} \\ (-1)^{n+1} & \text{if} \quad y_{2i} > x_{i+1}. \end{cases} \tag{4.8}$$

Note that $y_{2i} > x_{i+1}$ can happen only when $2i=2m=j^*$. In this case $n_{j^*}=1$ and, in view of (4.7), (4.8) is equivalent to

$$\operatorname{sign}\ u(y_1)\, v(x_1) = 1.$$

We set $N=D\backslash P$ and suppose, without loss of generality, that

$$\text{card } P \geqslant \text{card } N. \tag{4.9}$$

We shall show that $u-\varepsilon v$ has at least n sign changes on K contradicting the assumption on U to be a weak Chebyshev space. (If card $P<$ card N, then similar argumentation shows that $u+\varepsilon v$ has at least n sign changes.)

We first prove the following statement.

LEMMA. *The function $u-\varepsilon v$ has at least*

$$n-n_{j^*}+\text{card }(P\backslash\{j^*\})-\text{card }(N\backslash\{j^*\}) \tag{4.10}$$

sign changes in the interval $[y_{j^*+1}, y_{j^*}]$.

Proof. Suppose that $j\neq j^*$ and $n_j\geqslant 1$. Let

$$[y_j, y_{j+1}]\cap(T\cup X)=\{\zeta_1, ..., \zeta_{n_j}\}$$

such that $y_j<\zeta_1<\cdots<\zeta_{n_j}<y_{j+1}$. Since $u(\zeta_i)=0$, $i=1, ..., n_j$, it follows from (4.2) that $u-\varepsilon v$ has at least n_j-1 sign changes in $[\zeta_1, \zeta_{n_j}]$. Moreover, if $j\notin N$, we can find some additional sign changes of $u-\varepsilon v$ in $[y_j, y_{j+1}]$.

Indeed, if $j\in\{1, ..., 2m\}\backslash D$, then by the definition of D,

$$\text{sign } u(y_j)\, u(y_{j+1})=(-1)^{n_j}.$$

Therefore, in view of (4.6), we obtain

$$\text{sign}(u-\varepsilon v)(y_j)=(-1)^{n_j}\,\text{sign}(u-\varepsilon v)(y_{j+1})$$

which would be impossible if $u-\varepsilon v$ had exactly n_j-1 sign changes in $[y_j, y_{j+1}]$. Thus, $u-\varepsilon v$ has at least n_j sign changes there when $j\in\{1, ..., 2m\}\backslash D$.

We next consider the case when $j\in P$. Then, if j is an odd number, it follows from (4.2) that

$$\text{sign } u(y_j)=\text{sign } v(\zeta_1).$$

(Note that in this case $X\cap[y_j, y_{j+1}]=\varnothing$ and $t_{k_j}=\zeta_1$.) Otherwise, if j is even, it follows from (2.4) and the fact that $T\subset\tilde{Z}$ that $[y_j, y_{j+1}]\cap(T\cup X)=\{x_{i+1}\}$ where $j=2i$. Thus $\zeta_1=x_{i+1}$ and by (4.8), again

$$\text{sign } u(y_j)=\text{sign } v(\zeta_1).$$

Summarizing both cases, and by (4.6), we obtain

$$\operatorname{sign}(u-\varepsilon v)(y_j) = -\operatorname{sign}(u-\varepsilon v)(\zeta_1).$$

Therefore, $u-\varepsilon v$ has at least n_j sign changes in $[y_j, \zeta_{n_j}]$ if $j \in P$. Moreover, it follows from (4.7) that

$$\operatorname{sign}(u-\varepsilon v)(y_j) = (-1)^{n_j+1} \operatorname{sign}(u-\varepsilon v)(y_{j+1}).$$

Hence, $u-\varepsilon v$ cannot have exactly n_j sign changes in $[y_j, y_{j+1}]$. By the above arguments, it has at least n_j+1 sign changes there when $j \in P$.

Thus we have shown that $u-\varepsilon v$ has at least n_j-1, n_j or n_j+1 sign changes in $[y_j, y_{j+1}]$ if $j \in N$, $j \in \{1, ..., 2m\} \backslash D$ or $j \in P$, respectively. Taking into consideration that we had supposed that $j \neq j^*$ we conclude that $u-\varepsilon v$ has at least

$$\sum_{j \in N \backslash \{j^*\}} (n_j - 1) + \sum_{j \notin D \cup \{j^*\}} n_j + \sum_{j \in P \backslash \{j^*\}} (n_j + 1)$$

sign changes in $[y_{j^*+1}, y_{j^*}] = \bigcup_{j=1;\, j \neq j^*}^{2m} [y_j, y_{j+1}]$, which, in view of (4.1), implies (4.10) and completes the proof of the lemma. ▮

To finish the proof of Theorem 2.4 we have to show that $u-\varepsilon v$ has additional sign changes in the interval $[y_{j^*}, y_{j^*+1}]$ if necessary. To this end we consider several cases. In each case we show that $u-\varepsilon v$ has at least n sign changes on K contradicting the assumption on U to be a weak Chebyshev space.

Case 1. Assume that $n_{j^*} = 0$. Then $j^* \notin D$ which implies that $P \backslash \{j^*\} = P$, $N \backslash \{j^*\} = N$ and, in view of (4.9), the lemma immediately yields that $u-\varepsilon v$ has at least n sign changes.

Case 2. Assume that $n_{j^*} = 1$ and $j^* \in N$. Then

$$\operatorname{card}(P \backslash \{j^*\}) - \operatorname{card}(N \backslash \{j^*\}) \geqslant 1,$$

and, hence,

$$n - n_{j^*} + \operatorname{card}(P \backslash \{j^*\}) - \operatorname{card}(N \backslash \{j^*\}) \geqslant n$$

which implies that $u-\varepsilon v$ has at least n sign changes.

Case 3. Assume that $n_{j^*} = 1$ and $j^* \notin D$. By the lemma, $u-\varepsilon v$ has at least $n-1$ sign changes in $[y_{j^*+1}, y_{j^*}]$. Since $n_{j^*} \neq 0$, it follows from the definition of D (see (4.7)), that

$$\operatorname{sign} u(y_{j^*})\, u(y_{j^*+1}) = (-1)^{n_{j^*}+n+1}.$$

Hence, by (4.6) we obtain

$$\operatorname{sign}(u-\varepsilon v)(y_{j^*+1}) = (-1)^n \operatorname{sign}(u-\varepsilon v)(y_{j^*}).$$

Therefore, $u-\varepsilon v$ has at least n sign changes in $[y_{j^*+1}, y_{j^*}]$.

Case 4. Assume that $n_{j^*}=1$ and $j^* \in P$. It follows from the lemma that $u-\varepsilon v$ has at least $n-2$ sign changes in $[y_{j^*+1}, y_{j^*}]$. By (4.6) and (4.7) we obtain that

$$\operatorname{sign}(u-\varepsilon v)(y_{j^*}) = (-1)^{n+1} \operatorname{sign}(u-\varepsilon v)(y_{j^*+1}).$$

Therefore, $u-\varepsilon v$ must have at least $n-1$ sign changes in $[y_{j^*+1}, y_{j^*}]$. Since $n_{j^*}=1$, we have

$$(y_{j^*}, y_{j^*+1}) \cap (X \cup T) = \{\zeta\}.$$

It follows from (4.3), (4.4), (4.7) and (4.8) that

$$\operatorname{sign} v(\zeta) = \begin{cases} \operatorname{sign} u(y_{j^*}) & \text{if } \zeta > y_{j^*} \\ \operatorname{sign} u(y_{j^*+1}) & \text{if } \zeta < y_{j^*+1}. \end{cases}$$

Then, since $u=0$ on $X \cup T$, the function $u-\varepsilon v$ has a sign change in (y_{j^*}, ζ) if $\zeta > y_{j^*}$ and in (ζ, y_{j^*+1}) if $\zeta < y_{j^*+1}$, respectively. Anyway, $u-\varepsilon v$ has at least one sign change outside $[y_{j^*+1}, y_{j^*}]$.

Again, the total number of sign changes is at least n.

Case 5. Assume that $n_{j^*} \geqslant 2$. Then j^* must be odd which implies that $j^* = 2m-1$. It then follows that

$$T \cap [y_{2m-1}, y_{2m}] = \{t_{k_{2m-1}}, ..., t_{k_{2m-1}+n_{2m-1}-1}\},$$

and, for some $p \in \{0, ..., n_{2m-1}\}$,

$$t_{k_{2m-1}+p} < \cdots < t_{k_{2m-1}+n_{2m-1}-1} < y_{2m} < x_1$$
$$< \cdots < x_m < y_{2m-1} < t_{k_{2m-1}} < \cdots < t_{k_{2m-1}+p-1}.$$

We set

$$t_{\min} = \begin{cases} t_{k_{2m-1}+p} & \text{if } p \neq n_{2m-1}, \\ y_{2m} & \text{if } p = n_{2m-1}, \end{cases} \qquad t_{\max} = \begin{cases} t_{k_{2m-1}+p-1} & \text{if } p \neq 0, \\ y_{2m-1} & \text{if } p = 0. \end{cases}$$

In view of (4.3), it is easy to see that $u-\varepsilon v$ has at least $n_{2m-1}-p-1$ sign changes in $[t_{\min}, y_{2m}]$ and at least p sign changes in $[y_{2m-1}, t_{\max}]$ if $p \neq 0$. Moreover, by (4.4), $u-\varepsilon v$ has at least n_{2m-1} sign changes in $[t_{\min}, y_{2m}]$ if $p=0$.

If $j^* \notin D$, then by the lemma, $u-\varepsilon v$ has at least $n-n_{2m-1}$ sign changes in $[y_{2m}, y_{2m-1}]$. By the definition of D,

$$\operatorname{sign} u(y_{2m}) = (-1)^{n_{2m-1}+n-1} \operatorname{sign} u(y_{2m-1})$$

and, in view of (4.6),

$$\operatorname{sign}(u-\varepsilon v)(y_{2m}) = (-1)^{n_{2m-1}+n-1} \operatorname{sign}(u-\varepsilon v)(y_{2m-1}).$$

Therefore, $u-\varepsilon v$ must in fact have at least $n-n_{2m-1}+1$ sign changes in $[y_{2m}, y_{2m-1}]$.

Thus, by the above arguments, $u-\varepsilon v$ has at least $(n_{2m-1}-p-1)+p+(n-n_{2m-1}+1)=n$ sign changes in K.

Finally, let $j^* \in D$. Then $j^* \in P$ since j^* is odd. By the lemma, the function $u-\varepsilon v$ has at least $n-n_{2m-1}-1$ sign changes in $[y_{2m}, y_{2m-1}]$. As above, we deduce from (4.6) and (4.7) that

$$\operatorname{sign}(u-\varepsilon v)(y_{2m}) = (-1)^{n_{2m-1}+n} \operatorname{sign}(u-\varepsilon v)(y_{2m-1})$$

which shows that $u-\varepsilon v$ must have at least $n-n_{2m-1}$ sign changes in $[y_{2m}, y_{2m-1}]$. If now $p=0$ or $p=n_{2m-1}$, then by the above arguments, $u-\varepsilon v$ has at least $(n-n_{2m-1})+n_{2m-1}=n$ sign changes in $[t_{\min}, y_{2m-1}]$ or $[y_{2m}, t_{\max}]$, respectively. If $p \in \{1, ..., n_{2m-1}-1\}$, then $u-\varepsilon v$ has at least $(n-n_{2m-1})+(n_{2m-1}-p-1)+p=n-1$ sign changes in $[t_{\min}, t_{\max}]$. Moreover, in view of (4.3),

$$\operatorname{sign}(u-\varepsilon v)(t_{\min}) = (-1)^{p+n} \operatorname{sign} u(y_{2m-1}),$$

$$\operatorname{sign}(u-\varepsilon v)(t_{\max}) = (-1)^{p} \operatorname{sign} u(y_{2m-1}),$$

which implies that

$$\operatorname{sign}(u-\varepsilon v)(t_{\min}) = (-1)^{n} \operatorname{sign}(u-\varepsilon v)(t_{\max}).$$

Hence, $u-\varepsilon v$ must in fact have at least n sign changes in $[t_{\min}, t_{\max}]$. ∎

REFERENCES

1. C. de Boor and K. Höllig, Bivariate box splines and smooth pp functions on a three direction mesh, *J. Comput. Appl. Math.* **9** (1983), 13–28.
2. J. M. Carnicer and J. M. Peña, Spaces with almost strictly totally positive bases, *Math. Nachrichten* **169** (1994), 69–79.
3. J. M. Carnicer and J. M. Peña, Least supported bases and local linear independence, *Numer. Math.* **67** (1994), 289–301.
4. O. Davydov, A class of weak Chebyshev spaces and characterization of best approximations, *J. Approx. Theory* **81** (1995), 250–259.

5. O. Davydov, On almost interpolation, *J. Approx. Theory* **91** (1997), 398–418.
6. O. Davydov and M. Sommer, Interpolation and almost interpolation by weak Chebyshev spaces, *in* "Approximation Theory IX" (C. K. Chui and L. L. Schumaker, Eds.), Vanderbilt Univ. Press, Nashville, 1998, pp. 25–32.
7. O. Davydov, M. Sommer, and H. Strauss, On almost interpolation by multivariate splines, *in* "Multivariate Approximation and Splines" (G. Nürnberger, J. W. Schmidt, and G. Walz, Eds.), Birkhäuser, Basel, 1997, pp. 45–58.
8. O. Davydov, M. Sommer, and H. Strauss, Locally linearly independent systems and almost interpolation, in "Multivariate Approximation and Splines" (G. Nürnberger, J. W. Schmidt, and G. Walz, Eds.), Birkhäuser, Basel, 1997, pp. 59–72.
9. O. Davydov, M. Sommer, and H. Strauss, On almost interpolation and locally linearly independent bases, *East J. Approx.* **5** (1999), 67–88.
10. W. Li, "Continuous Selections for Metric Projections and Interpolating Subspaces," Verlag Peter Lang, Frankfurt, 1991.
11. G. Nürnberger, L. L. Schumaker, M. Sommer, and H. Strauss, Interpolation by generalized splines, *Numer. Math.* **42** (1983), 195–212.
12. L. L. Schumaker, "Spline Functions: Basic Theory," Wiley, New York, 1981.
13. M. Sommer and H. Strauss, Weak Descartes systems in generalized spline spaces, *Constr. Approx.* **4** (1988), 133–145.
14. M. Sommer and H. Strauss, Interpolation by uni- and multivariate generalized splines, *J. Approx. Theory* **83** (1995), 423–447.
15. B. Stockenberg, On the number of zeros of functions in a weak Tchebyshev-space, *Math. Z.* **156** (1977), 49–57.